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1981 J. Phys. A: Math. Gen. 14 1849

(http://iopscience.iop.org/0305-4470/14/8/012)

J. Phys. A: Math. Gen. 14 (1981) 1849–1861. Printed in Great Britain

On Hamiltonian structure of integrable equations under the group and matrix reductions

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Received 7 November 1980

Abstract. The Hamiltonian structure of (*a*) differential equations integrable by means of an arbitrary-order linear spectral problem under reductions to classical Lie algebras B_N, C_N, D_N and also of (*b*) equations associated with the matrix analogue of the linear Zakharov–Shabat linear problem is analysed.

1. Introduction

The Hamiltonian interpretation of the differential equations integrable by the inverse scattering method has been discussed in many papers, beginning with the papers of Gardner (1971), Zakharov and Faddeev (1974) and Faddeev (1980). It has been demonstrated by Zakharov and Faddeev that an infinite series of equations of the same Hamiltonian structure is connected with the Korteweg–de Vries equation (to be precise, with the associated linear spectral problem). This observation has been developed and generalised by Flaschka and Newell (1975) in the framework of the AKNS method (Ablowitz *et al* 1974). It turns out to be possible to analyse, from the same standpoint, the Hamiltonian structure of all equations integrable by the Zakharov–Shabat linear spectral problem. Another approach to Hamiltonian integrable equations has been developed in the papers of Gel'fand and Dikij (1977, 1978) for example.

Recently, the AKNS method has been extended to the matrix linear spectral problem of arbitrary order (Newell 1979, Kulish 1979, Konopelchenko 1980a, b, 1981). Among the corresponding integrable equations there are, in particular, the generalisations of the sine–Gordon equation to any classical Lie group. In the general position, all equations of this class, as was shown in Newell (1979) and Konopelchenko (1980b, 1981), are Hamiltonian ones.

What we wish to consider in the present paper is the natural group reductions of the general equations integrable by a linear spectral problem of arbitrary order, i.e. the reductions connected with 'embedding' of potentials into one of the classical Lie algebras B_N, C_N, D_N. It is shown that these reduced equations (particularly, generalisations of the sine–Gordon equation to the $SO(N, \mathbb{C})$ and $Sp(2N, \mathbb{C})$ ($N = 1, 2, 3, \ldots$) groups are Hamiltonian ones. The Poisson brackets are given. The Hamiltonian structure of a class of equations integrable by the matrix generalisation of the Zakharov–Shabat linear problem is analysed. And also, some reductions of these equations are considered.

0305-4470/81/081849+13$01.50

The paper is arranged as follows. The general form of the integrable equations and their group reductions are examined in § 2. The Hamiltonian structure of the reduced equations is discussed in § 3. Section 4 is devoted to the Hamiltonian structure of equations integrable by the matrix Zakharov–Shabat linear problem.

2. General form of integrable equations and group reductions

We shall consider differential equations integrable by the linear spectral problem

$$\frac{\partial \psi}{\partial x} = \mathrm{i}\lambda A\psi + \mathrm{i}P(x, t)\psi \tag{2.1}$$

where λ is the spectral parameter ($\lambda \in \mathbb{C}$), A is a constant matrix of order N and the 'potentials' $P(x, t)$ are $N \times N$ matrices. The general integrable equations are of the following form (Konopelchenko 1980b)

$$\frac{\partial P(x, t)}{\partial t} - \mathrm{i} \sum_{\alpha=1}^{r_A} \Omega_\alpha(L_A^+, t)[H_\alpha, P(x, t)] = 0 \tag{2.2}$$

where $\Omega_1(\lambda, t), \ldots, \Omega_{r_A}(\lambda, t)$ are arbitrary meromorphic functions, $r_A = \dim g_{0(A)} - 1$, $g_{0(A)}$ is the zero component of the Fitting decomposition of the algebra $\mathrm{gl}(N, \mathbb{C})$ with respect to A ($[A, g_{0(A)}] = 0$). Matrices H_α, $\alpha = 1, \ldots, r_A + 1$ form the basis of the subalgebra $g_{0(A)}$. For arbitrary $B \in \mathrm{gl}(N, \mathbb{C})$, $B_{0(A)}$ and $B_{F(A)}$ denote the projections B onto $g_{0(A)}$ and $g_{F(A)}$, respectively ($g_{F(A)}$ is the direct sum of non-zero root subspaces in the Fitting decomposition $\mathrm{gl}(N, \mathbb{C})$ with respect to A). Operator L^+ is of the form

$$L^+\Phi = \mathrm{i}\frac{\partial \Phi}{\partial x} + [P(x, t), \Phi(x)]_{F(A)} + \mathrm{i}\left[P(x, t), \int_{-\infty}^{x} \mathrm{d}y\, [P(y, t), \Phi(y)]_{0(A)}\right].$$

Equation (2.2) is written in the gauge $P_{0(A)} = 0$. A sense of this gauge is that the purely gauge (non-dynamical) degrees of freedom are excluded from $P(x, t)$ (Konopelchenko 1980b).

In what follows we restrict ourselves to the case of a diagonal matrix A. If all elements of A are different, then $g_{0(A)}$ is a set of all diagonal matrices and $r_A = N - 1$. In this case, when all $N^2 - N$ components of $P(x, t)(P = P_{F(A)})$ are independent, equation (2.2) is an equation on the algebra $\mathrm{gl}(N, \mathbb{C})$ (to be more precise, on $\mathrm{gl}(N, \mathbb{C})_F$) in the general position. In the general position, equations of the type (2.2) are Hamiltonian ones with the Poisson bracket (Newell 1979, Konopelchenko 1980b, 1981)

$$\{I, H\} = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{Tr}\left(\frac{\delta I}{\delta P(x, t)}\left[A, \frac{\delta H}{\delta P(x, t)}\right]\right). \tag{2.3}$$

Generally speaking, under the reductions of general equations the Hamiltonian structure varies (for $N = 2$ see Flaschka and Newell (1975)); for the reduction problem see e.g. Zakharov (1980), Zakharov and Shabat (1979).

Now, consider the natural group reductions for general equations (2.2), i.e. the reductions associated with transition from the algebra $\mathrm{gl}(N, \mathbb{C})$ to one of the classical Lie algebras A_N, B_N, C_N, D_N. The reduction $\mathrm{gl}(N, \mathbb{C}) \to \mathrm{sl}(N, \mathbb{C})$ keeps equation (2.2) in the general position. With the purpose of describing the remaining three non-trivial reductions, the choice of a definite matrix realisation of the algebras B_N, C_N, D_N seems to be necessary. In our paper we follow Bourbaki (1972), and

(i) we identify the algebra B_N with the algebra so$(2N+1, \mathbb{C})$ of the quadratic matrices P of order $2N+1$ for which

$$P_{\mathrm{T}} = -T_B P T_B^{-1} \tag{2.4}$$

where

$$T_B = \begin{pmatrix} 0 & 0 & s \\ 0 & -2 & 0 \\ s & 0 & 0 \end{pmatrix}$$

and s is the quadratic matrix of order N, all elements of which are zero ones except for those placed at the by-side diagonal and equal to unity†. The symbol T means the matrix transposition.

(ii) The algebra $C_N (N \geqslant 1)$ is identified with the algebra sp$(2N, \mathbb{C})$ of quadratic matrices P such that

$$P_{\mathrm{T}} = -T_C P T_C^{-1} \tag{2.5}$$

where

$$T_C = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}.$$

(iii) The algebra $D_N (N \geqslant 2)$ is identified with the algebra so$(2N, \mathbb{C})$ of quadratic matrices P for which

$$P_{\mathrm{T}} = -T_D P T_D^{-1}. \tag{2.6}$$

Such a realisation of the algebras B_N, C_N, D_N is suitable for our purposes, since it enables us to consider all three algebras simultaneously. The specific feature of each algebra will manifest itself only in an order of the matrices P and T (the odd order for B_N and the even order for C_N and D_N) and also in the form of the matrix $T(T_B, T_C, T_D)$. In all three cases, the Cartan subalgebras consist of diagonal matrices with the basis

$$\{H_\alpha, \alpha = 1, \ldots, N; H_\alpha = E_{\alpha\alpha} - E_{2N+1-\alpha, 2N+1-\alpha}\}$$

where

$$(E_{\alpha\beta})_{ik} = \delta_{\alpha i}\delta_{\beta k} \qquad (\alpha, \beta = 1, \ldots, N; i = 1, \ldots, 2N(2N+1)).$$

It is easy to be convinced, following Konopelchenko (1980b), that equations (2.2) admit reductions to the algebras so$(2N+1, \mathbb{C})$, sp$(2N, \mathbb{C})$, and so$(2N, \mathbb{C})$ described above, if $(Y = \sum_\alpha \Omega_\alpha(\lambda, t) H_\alpha)$

$$\begin{aligned} A_{\mathrm{T}} &= -TAT^{-1} = A \\ Y_{\mathrm{T}} &= -TYT^{-1} = Y \end{aligned} \tag{2.7}$$

that is

$$A = \sum_{\alpha=1}^{N} a_\alpha H_\alpha \qquad Y = \sum_{\alpha=1}^{N} \Omega_\alpha(\lambda, t) H_\alpha \tag{2.8}$$

† s is of the form $\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$.

where $\{H_\alpha, \alpha = 1, \ldots, N\}$ are the bases of the Cartan subalgebras of the algebras $\mathrm{so}(2N+1, \mathbb{C})$, $\mathrm{sp}(2N, \mathbb{C})$, $\mathrm{so}(2N, \mathbb{C})$; $a_1, \ldots, a_N$ are any numbers; $\Omega_1(\lambda, t), \ldots, \Omega_N(\lambda, t)$ are arbitrary functions λ. In this case,

$$T^{-1}\psi_{\mathrm{T}}(x, t; \lambda)T = \psi^{-1}(x, t; \lambda). \tag{2.9}$$

For the transition matrix of the linear spectral problem (2.1) under the reductions to the algebras $\mathrm{so}(2N+1, \mathbb{C})$, $\mathrm{sp}(2N, \mathbb{C})$, and $\mathrm{so}(2N, \mathbb{C})$, respectively, we have

$$T^{-1}S_{\mathrm{T}}(\lambda, t)T = S^{-1}(\lambda, t). \tag{2.10}$$

In a particular case $\Omega_\alpha = \omega_\alpha \lambda^{-1}$, where $\omega_\alpha (\alpha = 1, \ldots, N)$ are arbitrary numbers, equations (2.2) may be written as

$$\frac{\partial}{\partial t}\left(U^{-1}\frac{\partial U}{\partial x}\right)_F + [A, U^{-1}YU] = 0 \tag{2.11}$$

where

$$Y = \sum_{\alpha=1}^{N} \omega_\alpha H_\alpha \qquad U_{\mathrm{T}}(x, t)TU(x, t) = T.$$

Equations (2.11) represent a generalisation of the sine–Gordon equation to the groups $\mathrm{SO}(N, \mathbb{C})$, $\mathrm{Sp}(2N, \mathbb{C}) - U \in \mathrm{SO}(N, \mathbb{C})$, or $U \in \mathrm{Sp}(2N, \mathbb{C})$. (For the generalisations of the sine–Gordon equation to the groups $\mathrm{SU}(N)$ and $\mathrm{SO}(N)$ see Budagov and Takhtadjan (1977), Budagov (1978), Zakharov and Mikhailov (1978) and Konopelchenko (1980a).)

One emphasises that if all elements of the matrices $\mathrm{i}A$, $\mathrm{i}Y$ and $\mathrm{i}P$ are real, i.e. all a_α and $\Omega_\alpha(\lambda, t)$ in (2.8) are purely imaginary, equations (2.2) admit additional reductions to the algebras $\mathrm{so}(2N+1, \mathbb{R})$, $\mathrm{sp}(2N, \mathbb{R})$, $\mathrm{so}(2N, \mathbb{R})$. In particular, we have the generalisation (2.11) of the sine–Gordon equation to the groups $\mathrm{SO}(2N+1, \mathbb{R})$, $\mathrm{Sp}(2N, \mathbb{R})$ and $\mathrm{SO}(2N, \mathbb{R})$.

3. The Hamiltonian structure of equations under the reductions to the algebras B_N, C_N, D_N

Let us present a few formulae (see Konopelchenko 1980b) which will be required in the following. We denote the fundamental matrices solutions (2.1) $F^+ \xrightarrow[x\to\infty]{} \exp \mathrm{i}\lambda Ax$, $F^- \xrightarrow[x\to-\infty]{} \exp \mathrm{i}\lambda Ax$ by $F^+(x, t; \lambda)$ and $F^-(x, t; \lambda)$ (assuming that $P(x, t) \xrightarrow[|x|\to\infty]{} 0$) and the transition matrix $F^+(x, t; \lambda) = F^-(x, t; \lambda)S(\lambda, t)$ by $S(\lambda, t)$. For two matrices $P(x, t)$ and $P'(x, t)$ and the corresponding $S(\lambda, t)$, $F^+(x, t; \lambda)$, and $S'(\lambda, t)$, $F^{+\prime}(x, t; \lambda)$ we have the relation

$$S' - S = -\mathrm{i}S \int_{-\infty}^{+\infty} \mathrm{d}x\, F^{+-1}(P' - P)F^{+\prime}. \tag{3.1}$$

If it is assumed that

$$\frac{\mathrm{d}S(\lambda, t)}{\mathrm{d}t} = \mathrm{i}[Y(\lambda, t), S(\lambda, t)] \tag{3.2}$$

where Y is any element of the Cartan subalgebra containing A, it follows from (3.1) that

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ \mathrm{Tr}\left[\left(\frac{\partial P}{\partial t}-\mathrm{i}\sum_{\alpha=1}^{r_A}\Omega_\alpha(\lambda, t)[H_\alpha, P]\right)\phi^{++F}(x, t; \lambda)\right]=0 \tag{3.3}$$

where H_α, $\alpha=1, \ldots, r_A$ is the basis of the Cartan subalgebra, $Y=\Sigma_{\alpha=1}^{r_A}\Omega_\alpha(\lambda, t)H_\alpha$ and $\overset{++}{\phi}{}^{(in)}_{kl}(x, t; \lambda)=(F^{+-1})_{il}(F^{+})_{kn}$. Then, the following relation holds

$$L\overset{++}{\phi}{}^{(in)}_{F(A)}=\lambda[A, \overset{++}{\phi}{}^{(in)}_{F(A)}]+[P(x, t), \overset{++}{\phi}{}^{(in)}_{0(A)}(+\infty)] \tag{3.4}$$

where

$$L\Phi=-\mathrm{i}\frac{\partial\phi}{\partial x}-[P, \phi]_{F(A)}+\mathrm{i}\left[P(x, t), \int_x^{\infty}\mathrm{d}y\ [P(y, t), \phi(y)]_{0(A)}\right] \tag{3.5}$$

and, finally, taking into account the equality $\Omega(\lambda)\overset{++}{\phi}{}^{+(in)}_{F(A)}=\Omega(L_A)\overset{++}{\phi}{}^{(in)}_{F(A)}$ (where $[A, \psi_A]=\psi$; $i\neq n$) and proceeding, in (3.3), from the operator L to the operator L^+ adjoint to it with respect to the bilinear form $\int_{-\infty}^{\infty}\mathrm{d}x\ \mathrm{Tr}(\phi(x)\psi(x))$, we obtain equations (2.2).

In the general case, the Hamiltonian structure of equations (2.2) is proved, using the following relation resulting from (3.1):

$$\delta S(\lambda, t)=-\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}x\sum_{kl}\delta P_{kl}(x, t)\overset{-+}{\phi}{}^{(in)}_{lk}(x, t; \lambda) \tag{3.6}$$

where $\overset{-+}{\phi}{}^{(in)}_{kl}=(F^{--1})_{il}(F^{+})_{kn}$, and also in the equality (Konopelchenko 1980b)

$$L^+\overset{-+}{\phi}{}^{(in)}_{F(A)}=-\lambda[A, \overset{-+}{\phi}{}^{(in)}_{F(A)}]+[\overset{-+}{\phi}_{0(A)}(-\infty), P(x, t)] \tag{3.7}$$

where $\overset{-+}{\phi}{}^{(in)}_{ml}(-\infty)=\delta_{il}S_{mn}$.

In the Hamiltonian interpretation of equations (2.2) in the general case the fact is significant that all $P_{kl}(x, t)$ are independent dynamical variables. Under the reductions of the general equations (2.2) we have certain relations between the variables $P_{kl}(x, t)$. In our case of the reductions to the algebras B_N, C_N, D_N, they are of the form

$$P_{\mathrm{T}}=-TPT^{-1}. \tag{3.8}$$

Following the standard procedure, it is necessary to resolve these constraints, i.e. to introduce the set $Q(x, t)$ of independent dynamical variables. One can parametrise the set of matrices $P(x, t)$ satisfying relations (3.8) by various ways. We introduce independent dynamical variables as follows. Let us represent $P(x, t)$ in the form

$$P=Q-T^{-1}Q_{\mathrm{T}}T \tag{3.9}$$

where $Q(x, t)$ is the left-triangular matrix, i.e. the matrix all elements of which placed below the by-side diagonal are equal to zero†. It is not hard to convince oneself, using the expressions for the matrices $P(x, t)$ satisfying (3.8) (see Bourbaki 1972, ch 8, Zakharov 1980) that all elements of the matrix are independent and that formula (3.9) gives the general form of the matrices $P(x, t)$ belonging to the algebras B_N, C_N, D_N. The

† We would like to recall that our gauge is $P_{0(A)}=0$, i.e. $P_D\equiv\mathrm{diag}\,P=0$.

number of the elements Q coincides with the dimensionality of the corresponding algebra and is equal to so$(2N+1, \mathbb{C})-N(2N+1)$, for sp$(2N, \mathbb{C})-N(2N+1)$, $N(2N+1)$ for sp$(2N, \mathbb{C})$ and $N(2N-1)$ for so$(2N, \mathbb{C})$. It is necessary to emphasise the fact that for the orthogonal algebras so$(N, \mathbb{C})$ the elements of the matrix $Q(x, t)$ placed on the by-side diagonal are equal to zero.

Let us denote the operation of projection onto the left-triangular matrices by the symbol ∇, in particular $Q = Q_\nabla$.

Now, convert equation (2.2) to such a form in which the latter contains independent variables Q only. Let us start with equation (3.3). Following from the definition (3.9) and using the properties of the matrix trace (in particular $\mathrm{Tr}(Q\chi) = \mathrm{Tr}(Q\chi_\nabla)$), we get from (3.3) ($r_A = N$)

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{Tr}\left[\left(\frac{\partial Q}{\partial t} - \mathrm{i}\sum_{\alpha=1}^{N} [H_\alpha, Q]\Omega_\alpha(\lambda, t)\right) \overset{++}{\chi}{}^F_\nabla(x, t; \lambda)\right] = 0 \tag{3.10}$$

where $\chi^{(in)} = \phi_F^{(in)} - T^{-1}\phi_F^{(in)}T$. From (3.4) and (3.5) we find

$$\frac{\partial \overset{++}{\chi}{}^F}{\partial x} = \mathrm{i}\lambda[A, \overset{++}{\chi}{}^F] + \left[P(x), \int_x^{\infty} \mathrm{d}y [P(y), \overset{++}{\chi}{}^F]_D\right] + \mathrm{i}[P, \overset{++}{\chi}{}^F]_F. \tag{3.11}$$

Applying the operation ∇ to (3.11), we obtain

$$L_{(Q)A} \overset{++}{\chi}{}^F_{F\nabla} = \lambda \overset{++}{\chi}{}^F_{F\nabla} \tag{3.12}$$

where ($P = Q - T^{-1}Q_\mathrm{T}T$)

$$\begin{aligned} L_{(Q)}\chi_\nabla = & -\mathrm{i}\frac{\partial \chi}{\partial x} - [P, \chi_\nabla] + (T^{-1}[P, \chi_\nabla]_\mathrm{T} T)_{F\nabla} \\ & + \mathrm{i}\left[Q(x), \int_x^{+\infty} \mathrm{d}y ([P(y), \chi_\nabla(y)]_D - T^{-1}[P(x), \chi_\nabla(y)]_D T)\right]. \end{aligned} \tag{3.13}$$

As a result, equation (3.10) may be written in the following form

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{Tr}\left[\left(\frac{\partial Q}{\partial t} - \mathrm{i}\sum_{\alpha=1}^{N} [H_\alpha, Q(x)]\Omega_\alpha(L_{(Q)A})\right) \overset{++}{\chi}{}^F_\nabla\right] = 0. \tag{3.14}$$

Finally, coming in (3.14) from the operator $L_{(Q)}$ to the operator $L^+_{(Q)}$ adjoint to $L_{(Q)}$ with respect to the bilinear form $\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{Tr}(Q(x)\chi(x))$, we obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{Tr}\left[\overset{++}{\chi}{}^F_\nabla(x, t; \lambda)\left(\frac{\partial Q}{\partial t} - \mathrm{i}\sum_{\alpha=1}^{N} \Omega_\alpha(L^+_{(Q)A}, t)[H_\alpha, Q]\right)\right] = 0 \tag{3.15}$$

where

$$\begin{aligned} L^+_{(Q)}\chi = & \ \mathrm{i}\frac{\partial \chi}{\partial x} + [P, \chi]_\nabla + (T^{-1}[P, \chi]_\mathrm{T} T)_\nabla \\ & + \mathrm{i}\left[Q(x), \int_{-\infty}^{x} \mathrm{d}y ([P(y), \chi(y)]_D + T^{-1}[P(y), \chi(y)]_D T)\right]. \end{aligned} \tag{3.16}$$

Equation (3.15) is fulfilled, if

$$\frac{\partial Q(x, t)}{\partial t} - \mathrm{i}\sum_{\alpha=1}^{N} \Omega_{\alpha\nabla}(L^+_{(Q)A}, t)[H_\alpha, Q] = 0. \tag{3.17}$$

Equation (3.17) is the form of equation (2.2) containing the variables $Q(x, t)$ only. Note that equation (3.17) may be derived from equation (2.2) directly, applying the operation ∇.

Our next step is to show that equations of the form (3.17) are Hamiltonian ones. For this purpose, let us use relations (3.6) and (3.7). It follows from them that

$$\delta S_{in} = -\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x \sum_{kl} \delta Q_{kl} \overset{-+}{\chi}{}^{(in)}_{kl} \tag{3.18}$$

and

$$L^{+}_{(Q)} \overset{-+}{\chi}{}^{(in)} = -\lambda[A, \overset{-+}{\chi}{}^{(in)}] + [\overset{-+}{\phi}{}^{(in)}_{D}(-\infty), Q(x, t)]. \tag{3.19}$$

Let us introduce a quantity $\Pi_\alpha(x, t; \lambda)$:

$$(\Pi_\alpha(x, t; \lambda))_{kl} = \sum_{n=1}^{2N} \frac{(H_\alpha)_{nn}}{S_{nn}} \overset{-+}{\chi}{}^{(nn)}_{kl}. \tag{3.20}$$

For the algebra $\mathrm{so}(2N+1, \mathbb{C})$, n takes the values $1, 2, \ldots, 2N+1$. It follows from (3.18) ($S_{\mathrm{D}} = \operatorname{diag} S$) that

$$\Pi_\alpha(x, t; \lambda) = \mathrm{i}\frac{\delta}{\delta Q_{\mathrm{T}}} \operatorname{Tr}(H_\alpha \ln S_D(\lambda)) \tag{3.21}$$

and from (3.19) that

$$-(L^{+}_{(Q)A} - \lambda)[A, \Pi_\alpha] = [H_\alpha, Q(x, t)]. \tag{3.22}$$

Expanding the left- and right-hand sides of (3.22) in the asymptotic series of λ^{-1}, we obtain

$$(L^{+}_{(Q)A})^m [H_\alpha, Q(x, t)] = [A, \Pi^{(m+1)}_\alpha(x, t)] \qquad (m = 1, 2, 3, \ldots) \tag{3.23}$$

where $\Pi_\alpha(x, t; \lambda) = \Sigma_{m=0}^{\infty} \lambda^{-m} \Pi^{(m)}_\alpha(x, t)$. From (3.21) we find

$$\Pi^{(m)}_\alpha(x, t) = \mathrm{i}\frac{\delta}{\delta Q_{\mathrm{T}}} \operatorname{Tr}(H_\alpha C^{(m+1)}) \tag{3.24}$$

where $\ln S_D(\lambda) = \Sigma_{m=0}^{\infty} \lambda^{-m} C^{(m)}$ and $C^{(m)}$ ($m = 0, 1, 2, \ldots$) are the integrals of motion of equations (2.2) and (3.17), respectively (Konopelchenko 1980b).

It follows from relations (3.23) and (3.24) that equation (3.17) with $\Omega_\alpha(\lambda, t) = \Sigma_{m=0}^{\infty} \omega^{\alpha}_{m}(t)\lambda^m$ (ω^{α}_{m} are any numbers) is of the form

$$\frac{\partial Q(x, t)}{\partial t} = \left[A, \frac{\delta \mathscr{H}}{\delta Q_{\mathrm{T}}(x, t)}\right] \tag{3.25}$$

where

$$\mathscr{H} = \sum_{\alpha=1}^{N} \sum_{m=0}^{\infty} \omega^{\alpha}_{m}(t) \operatorname{Tr}(H_\alpha C^{(m+1)}). \tag{3.26}$$

It is easy to see that equation (3.25) may be written in the Hamiltonian form

$$\frac{\partial P}{\partial t} = \{P, \mathscr{H}\}$$

with the Hamiltonian (3.26) and the Poisson bracket

$$\{I(Q), \mathscr{H}(Q)\} = \int_{-\infty}^{+\infty} \mathrm{d}x \operatorname{Tr}\left(\frac{\delta I}{\delta Q}\left[A, \frac{\delta \mathscr{H}}{\delta Q}\right]\right). \tag{3.27}$$

The Hamiltonian structure of equations (3.17) with singular functions $\Omega_\alpha(\lambda, t) = \sum_{m=1}^{\infty} \omega_m^\alpha(t)(\lambda - \lambda_{0m})^{-m}$ is proved in a similar way. The Hamiltonian of such an equation is equal to

$$\mathscr{H} = \sum_{\alpha=1}^{N} \sum_{m=1}^{\infty} \omega_m^\alpha(t) \frac{(-1)}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial \lambda^{m-1}} \operatorname{Tr}(H_\alpha \ln S_{\mathrm{D}}(\lambda)) \right)_{\lambda=\lambda_{0m}} \tag{3.28}$$

and the Poisson bracket is given by formula (3.27).

In particular, equations (3.17) with $\mathrm{i}\Omega_\alpha = \omega_\alpha \lambda^{-1}$ (where ω_α are constants) which are equivalent to generalisations of (2.11) of the sine–Gordon equations to the groups $\mathrm{SO}(N, \mathbb{C})$ and $\mathrm{Sp}(2N, \mathbb{C})$ are Hamiltonian ones. The Hamiltonian is equal to $\mathscr{H} = \operatorname{Tr}(Y \ln S_{\mathrm{D}}(0))$ and $Q = (U^{-1}\partial U/\partial x)_{\nabla}$.

The Hamiltonian structure of equations of the type (2.2) under the reduction to the algebra $\mathrm{so}(N, \mathbb{C})$ has also been examined in Konopelchenko (1981). The basis has been chosen in such a way that $P_{\mathrm{T}} = -P$. The associated Poisson bracket is greatly distinguished from (3.27): its kernel contains the operator of the covariant derivative type. This difference is due to a different choice of the coordinates in a phase space.

In conclusion, it is worth noting that just as in the general case (Konopelchenko 1980b, 1981), under the reductions to the algebras $\mathrm{so}(N, \mathbb{C})$, $\mathrm{sp}(2N, \mathbb{C})$ an infinite series of symplectic structures corresponds to equations (3.17). The Poisson brackets are obtained by introducing into the kernel of the bracket (3.27) any degrees of the operator $\overset{+}{L}_{(Q)A}$ (for the hierarchy of the Poisson brackets at $N = 2$ see Kulish and Reiman (1978)).

4. Hamiltonian structure of equations integrable by a matrix generalisation of the Zakharov–Shabat linear problem

Let us now turn to the linear problem (2.1) of order $2N$ with the matrix $A = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ where I is the unit matrix of order N. In the gauge $P_{0(A)} = 0$ the linear problem (2.1) is reduced to

$$\frac{\partial \psi}{\partial x} = \mathrm{i}\lambda \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \psi + \mathrm{i} \begin{pmatrix} 0 & Q(x, t) \\ R(x, t) & 0 \end{pmatrix} \psi \tag{4.1}$$

where† $Q(x, t)$, $R(x, t)$ are quadratic matrices of the Nth order and 0 is the zero matrix of order N. The problem (4.1) is the matrix analogue of the well known Zakharov–Shabat linear spectral problem ($N = 1$).

The equations integrable by means of (4.1) are characterised, in the general case, by $2N^2 - 1$ arbitrary functions (Konopelchenko 1980b). Among them there are equations of the form

$$\frac{\partial P}{\partial t} - 2\mathrm{i}\Omega(\overset{+}{L}_A, t) A P = 0 \tag{4.2}$$

where $P = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$, $\Omega(\lambda, t)$ is the arbitrary meromorphic function and

$$\overset{+}{L}_A \phi = \frac{\mathrm{i}}{2} \frac{\partial \phi}{\partial x} A + \frac{\mathrm{i}}{2} \left[P(x), \int_{-\infty}^{x} \mathrm{d}y \, [P(y), \phi(y) A] \right]. \tag{4.3}$$

† In § 3 the matrix Q means another quantity; for this reason, it is ruled out from § 4.

Since in our case $[P, Q_F]_F = 0$, the general operator L_Λ^+ reduces to (4.3). Equations (4.2) are matrix analogues of the equations examined in Flaschka and Newell (1975) and Ablowitz *et al* (1974). At $\Omega(\lambda) = -2\lambda^2$ we have the system of matrix equations:

$$\mathrm{i}\frac{\partial Q}{\partial t}+\frac{\partial^2 Q}{\partial x^2}+2QRQ=0$$

$$\mathrm{i}\frac{\partial R}{\partial t}-\frac{\partial^2 R}{\partial x^2}-2RQR=0.$$

Under the reduction $R = \pm Q^+$ we obtain the matrix analogue of the Schrödinger nonlinear equation (NLS). If $\Omega(\lambda) = -4\lambda^3$, equations (4.2) are of the form

$$\frac{\partial Q}{\partial t}+\frac{\partial^3 Q}{\partial x^3}+3\frac{\partial Q}{\partial x}RQ+3QR\frac{\partial Q}{\partial x}=0$$

$$\frac{\partial R}{\partial t}+\frac{\partial^3 R}{\partial x^3}+3\frac{\partial R}{\partial x}QR+3RQ\frac{\partial R}{\partial x}=0.$$

Under the reductions $R=\alpha Q$ and $R=1$ we obtain the matrix analogues of the modified Korteweg–de Vries equations (MKDV) and the Korteweg–de Vries (KDV) equation, respectively. At $\Omega(\lambda)\sim\lambda^{-1}$ and $R=-Q$ we have the matrix analogue of the sine–Gordon equation. It is worth mentioning that the matrix analogues of NLS, KDV, MKDV have been considered by Zakharov (1979) and Marchenko (1977).

Let us come now to the Hamiltonian structure of equations of the form (4.2). It is clear that in the general position they are Hamiltonian ones and the Poisson bracket is given by formula (2.3). In variables Q and R this bracket is of the form

$$\{I, \mathscr{H}\}=2\int_{-\infty}^{+\infty}\mathrm{d}x\ \mathrm{Tr}\left(\frac{\delta I}{\delta R}\frac{\delta\mathscr{H}}{\delta Q}-\frac{\delta I}{\delta Q}\frac{\delta\mathscr{H}}{\delta R}\right). \tag{4.4}$$

Under the reduction $R = \pm Q^+$ the bracket (4.4) is conserved.

A non-trivial modification of the symplectic structure appears under the reductions $R=\alpha Q$ (α is an arbitrary non-zero number) which take place at any odd functions $\Omega(\lambda)$. Indeed, it is easy to see that if $R=\alpha Q$, then the bracket (4.4) becomes degenerate (i.e. $\{I, \mathscr{H}\}_{(4.4)}=0$ for any I and $\mathscr{H}$). So, one must project the equations (4.2) onto the submanifold of the independent dynamical variables $Q(x, t)$. Then, it is necessary to investigate the Hamiltonian structure of these reduced equations.

Let us rewrite equations (4.7) in the form containing Q only.

From equation (3.3) we obtain ($\phi_F=\left(\begin{smallmatrix}0 & \phi_2\\ \phi_3 & 0\end{smallmatrix}\right)$)

$$\int_{-\infty}^{+\infty}\mathrm{d}x\ \mathrm{Tr}\left\{\frac{\partial Q}{\partial t}\overset{++}{\chi}-2\mathrm{i}Q\Omega(\lambda)\overset{++}{\psi}\right\}=0 \tag{4.5}$$

where $\psi=\phi_3-\alpha\phi_2$, $\chi=\phi_3+\alpha\phi_2$.

We have to find an operator L such that $L\overset{++}{\psi}=\lambda\overset{++}{\psi}$. To do this, present the equations which are satisfied by $\overset{++}{\psi}$ and $\overset{++}{\chi}$. From (3.4) we obtain

$$\mathrm{i}D_+\overset{++}{\psi}=2\lambda\overset{++}{\chi} \tag{4.6}$$

$$\mathrm{i}D_-\overset{++}{\chi}=2\lambda\overset{++}{\psi} \tag{4.7}$$

where

$$D_+ = \frac{\partial}{\partial x} - \alpha\left[Q(x), \int_x^{+\infty} \mathrm{d}y\, [Q(y), \cdot]_+\right]_+$$
$$D_- = \frac{\partial}{\partial x} - \alpha\left[Q(x), \int_x^{+\infty} \mathrm{d}y\, [Q(y), \cdot]_-\right]_- \tag{4.8}$$

where $[,]_+$, $[,]_-$ denote the anticommutator and commutator, respectively. Substituting χ from (4.6) into (4.7), we have

$$L_{(Q)} \overset{++}{\psi} = \lambda^2 \overset{++}{\psi} \tag{4.9}$$

where

$$L_{(Q)}\psi = -\tfrac{1}{4}D_-D_+\psi. \tag{4.10}$$

Further, let us transform the first term in (4.5) into the form containing ψ, instead of χ. Let us introduce the matrix $w(x, t)$ of order N such that

$$\frac{\partial Q}{\partial t} = D_- W. \tag{4.11}$$

Bearing in mind that

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}\left(\frac{\partial Q}{\partial t}\chi\right) = -\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}(WD_-\chi)$$

using (4.7) and (4.9), and also assuming that $W(\pm\infty) = 0$ we transform (4.5) into the form

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}(W(x) \overset{++}{\psi}(x) - Q(x, t)\omega(L_{(Q)}, t) \overset{++}{\psi}(x)) = 0 \tag{4.12}$$

where $\omega(\lambda^2) = \lambda^{-1}\Omega(\lambda)$.

From (4.12) we get

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}[\overset{++}{\psi}(x, t)(W(x) - \omega(L^+_{(Q)}, t)Q(x, t))] = 0 \tag{4.13}$$

where $L^+_{(Q)}$ is the operator adjoint to $L_{(Q)}$ with respect to the bilinear form $\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}(Q(x)\psi(x))$. It is equal to

$$L^+_{(Q)} = -\tfrac{1}{4}D^+_{(+)}D^+_{(-)} \tag{4.14}$$

where

$$D^+_{(\pm)} = \frac{\partial}{\partial x} + \alpha\left[Q(x), \int_{-\infty}^{x} \mathrm{d}y\, [Q(y), \cdot]_\pm\right]_\pm. \tag{4.15}$$

Equality (4.13) is fulfilled, if

$$W(x, t) - \omega(L^+_{(Q)}, t)Q(x, t) = 0.$$

Taking into account (4.11), we find

$$\frac{\partial Q(x, t)}{\partial t} - D_-\omega(L^+_{(Q)}, t)Q(x, t) = 0. \tag{4.16}$$

Equation (4.16) is a form of equation (4.7) (at $R = \alpha Q$) which contains the independent dynamical variables Q only.

We shall attempt now to prove the Hamiltonian character of equations (4.16). From (3.6) we have

$$\overset{-+}{\chi}{}^{(in)}(x,t) = \mathrm{i}\frac{\delta S_{in}}{\delta Q_{\mathrm{T}}(x,t)}. \tag{4.17}$$

Making use of the analogues of equation (4.5) for $\overset{-+}{\phi}{}^{(in)}_F$, we obtain

$$\begin{aligned} \mathrm{i}D_+ \sum_{n=1}^{2N} A_{nn}\frac{\overset{+}{\psi}{}^{(nn)}}{S_{nn}} &= 2\lambda \sum_{n=1}^{2N} A_{nn}\frac{\overset{-+}{\chi}{}^{(nn)}}{S_{nn}} - 4\alpha Q \\ \mathrm{i}D_- \sum_{n=1}^{2N} A_{nn}\frac{\overset{-+}{\chi}{}^{(nn)}}{S_{nn}} &= 2\lambda \sum_{n=1}^{2N} A_{nn}\frac{\overset{-+}{\psi}{}^{(nn)}}{S_{nn}}. \end{aligned} \tag{4.18}$$

Hence,

$$L^{+}_{(Q)}\Pi(x,t;\lambda) = \lambda^2\Pi(x,t;\lambda) - 2\alpha\lambda Q(x,t) \tag{4.19}$$

where

$$\Pi(x,t;\lambda) = \sum_{n=1}^{2N} \frac{\overset{-+}{\chi}{}^{(nn)}(x,t;\lambda)}{S_{nn}(\lambda)}$$

and from (4.17)

$$\Pi(x,t;\lambda) = \mathrm{i}\frac{\delta}{\delta Q_{\mathrm{T}}(x,t)}\mathrm{Tr}(A \ln S_D(\lambda)). \tag{4.20}$$

Writing (4.19) in the form $\Pi(x,t;\lambda)/2\alpha\lambda = (\lambda^2 - L^{+}_{(Q)})^{-1}Q(x,t)$ and expanding the left- and right-hand sides in asymptotic series of λ^{-1}, we obtain

$$(L^{+}_{(Q)})^n Q = \frac{1}{2\alpha}\Pi^{(2n+1)}(x,t) \tag{4.21}$$

where

$$\Pi(x,t;\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n}\Pi^{(n)}(x,t).$$

From equalities (4.20) and (4.21) we have

$$(L^{+}_{(Q)})^n Q = \frac{\mathrm{i}}{2\alpha}\frac{\delta}{\delta Q_{\mathrm{T}}}\mathrm{Tr}(AC^{(2n+1)}) \qquad n = 1, 2, \ldots \tag{4.22}$$

where $C^{(n)}$ are the integrals of motion ($\ln S_D(\lambda) = \Sigma_{n=0}^{\infty} \lambda^{-n}C^{(n)}$).

As a result, equation (4.16) with any entire function $\omega(\lambda^2) = \Sigma_{m=0}^{\infty} \omega_m\lambda^{2m}$ is written as

$$\frac{\partial Q}{\partial t} - D_-\frac{\delta \mathscr{H}}{\delta Q_{\mathrm{T}}} = 0 \tag{4.23}$$

where

$$\mathscr{H} = \frac{\mathrm{i}}{2\alpha}\sum_{m=1}^{\infty} \omega_m \,\mathrm{Tr}(AC^{(2m+1)}). \tag{4.24}$$

There is no difficulty in seeing that equations (4.23) may be represented in the Hamiltonian form $\partial Q/\partial t = \{Q, \mathscr{H}\}$ with the Hamiltonian $\mathscr{H}$ (4.24) and Poisson bracket

$$\{I(Q), \mathscr{H}(Q)\} = \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{Tr}\left(\frac{\delta I}{\delta Q_{\mathrm{T}}} D_{-} \frac{\delta \mathscr{H}}{\delta Q_{\mathrm{T}}}\right). \tag{4.25}$$

One can examine equations (4.16) with singular functions of the form $\omega(\lambda^2)$ in an analogous way:

$$\omega(\lambda^2) = \sum_{m=1}^{\infty} \omega_m (\lambda^2 - \lambda_{0m}^2)^{-m} \tag{4.26}$$

where ω_m $(m = 1, 2, \ldots)$ are arbitrary numbers. It follows from (4.19) and (4.20) that

$$(L_{(Q)}^{+} - \lambda^2)^{-m} Q(x, t) = -\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial(\lambda^2)^{m-1}} \frac{1}{2\alpha} \frac{\Pi(x, t; \lambda)}{\lambda}. \tag{4.27}$$

Using (4.27) and (4.20), equations (4.16) with $\omega(\lambda^2)$ of the type (4.26) may be represented in the form

$$\frac{\partial Q}{\partial t} + \sum_{m=0}^{\infty} D_{-} \frac{\mathrm{i}\omega_m}{2\alpha(m-1)!} \frac{\delta}{\delta Q_{\mathrm{T}}} \left(\frac{\partial^{m-1}}{\partial(\lambda^2)^{m-1}} \left(\frac{\mathrm{Tr}(A \ln S_D(\lambda))}{\lambda} \right) \right)_{\lambda = \lambda_{0m}} = 0. \tag{4.28}$$

These equations are Hamiltonian ones with respect to the Poisson bracket (4.25) with the Hamiltonian

$$\mathscr{H} = -\sum_{m=1}^{\infty} \frac{\omega_m}{2\alpha(m-1)!} \left(\frac{\partial^{m-1}}{\partial(\lambda^2)^{m-1}} \left(\frac{\mathrm{Tr}(A \ln S_D(\lambda))}{\lambda} \right) \right)_{\lambda^2 = \lambda_{0m}^2}. \tag{4.29}$$

In particular, the Hamiltonian of the matrix generalisation of the sine–Gordon equation $(\omega = \lambda^{-2}, \alpha = -1)$ is equal to $\mathscr{H} = \frac{1}{2} (\partial/\partial\lambda) \, \mathrm{Tr}(A \ln S_D(0))$.

Thus, we have shown that equations of the form (4.2) which are integrable by the spectral problem (4.1) are Hamiltonian ones in the general position and also under the reductions $R = \pm Q^{+}$, $R = \alpha Q$. Note that the kernel of the Poisson bracket (4.25) contains the integro-differential operator D_{-}.

With $N = 1$, $Q(x, t)$ is the numerical function, operator $D_{-} = \partial/\partial x$ and in the case $\alpha = -1$ formulae (4.15)–(4.29) are converted to the corresponding formulae of Flaschka and Newell (1975).

Just as in the case $N = 1$ (Kulish and Reiman 1978), an infinite series of symplectic structures corresponds to equation (4.7).

In conclusion, we should like to mention that the Hamiltonian structure of equations integrable by the matrix spectral problem $-\partial^2\psi/\partial x^2 + q(x, t)\psi = \lambda^2\psi$, which is equivalent to (4.1) under the reduction $Q = \mathrm{i}q$, $R = \mathrm{i}$, has been examined in Kulish (1980).

Acknowledgment

The authors are indebted to P P Kulish who drew their attention to the latter paper.

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